Indian Statistical Institute
Mid-Semestral Examination 2007-2008
B.Math (Hons.) I Year
ANALYSIS II

Time: 3 hrs | Date:03/03/2008 | Marks : 40
Total : 40

1. For $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ in R^n, Let $\|\underset{\sim}{x}\|^2 = \sum_j x_j^2$.
(a) Let $M : R^n \to R^k$ be linear map. For $\underset{\sim}{u}$ in R^n. Prove that
$\|M\underset{\sim}{u}\| \leq \|M\|\|\underset{\sim}{u}\|$ [3].
(b) Let $R^n \xrightarrow{M} R^k \xrightarrow{L} R^p$ be linear maps. Show that
$$\|L \circ M\| \leq \|L\|\|M\|.$$
[3]

2. Let $F : M_{n\times n}(R) \to M_{n\times n}(R)$ be given by $F(A) = A^3$. Show that F has total derivative and
$$F'(A)(H) = A^2H + AHA + HA^2.$$
[4]

3. Let $x_1, x_2.....$ and $y_1, y_2.....$ be sequences of of real numbers such that $\Sigma_j(|x_j|^2 + |y_j|^2)$ is finite.

Show that $(\sum_j |x_jy_j|)^2 \leq (\sum_j |x_j|^2)(\sum_k |y_k|^2)$. [4]

4. Let $F : R^2 \to R$ be a function such that each of the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and continuous. Show that f has total derive. [5]

5. Let $f : [a, b] \to R$ be any bounded function. For any partition
$$P = \{a = x_0 < x_1 < x_2 < ...... < x_n = b\}$$
and $Y = Y[P] = \{y_0, y_1......y_{n-1}\}$ where $y_j \in [x_j, x_{j+1}], define$ $S(P, Y(P), f) = \sum_i f(y_i)l[x_i, x_{i+1}]$ where l is the length function. Assume that there exists a real Z such that for each $\epsilon > 0$ we can find P_ε so that for any partition $Q \supset P_\epsilon$ and any $Y(Q)$,
$$|S(Q, Y(Q), f) - Z| \leq \epsilon$$
(a) Show that f is Riemann inteprable. [3]
(b) Show that $\int_a^b f = Z$. [3]

6. Let Π_1, Π_2 be the planes in R^3 given by

$$\Pi_1 = \{a_1x_1 + a_2x_2 + a_3x_3 + a_0 = 0\},$$

$$\Pi_2 = \{b_1x_1 + b_2x_2 + b_3x_3 + b_0 = 0\}.$$

   Let the line L be the intersection of Π_1 and Π_2.By Lagranges method find dist $((0,0,0), L)$. [6]

7. Show that

$$f(x,y,z) = 4x + 2y - x^4y^2 + z^4$$

   has a critical point but no relative extreme points. [6]

8. Let $f : [0,1] \to R$ be given by
   $f(rational) = 1$
   $f(irrational) = -1$

   (a) Show how that f is not Riemann integrable. [2]
   (b) Let $g(x) = |f(x)|$ for all x. Is g Riemann integrable.?
   If so find $\int_0^1 g$ [1]